

Singular Gauge Transformation in Non-Commutative $U(2)$ Gauge Theory

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Abstract

A method developed by Polychronakos to study singular gauge transformations in 1+2 dimensional non-commutative Chern-Simons gauge theory is generalized from $U(1)$ group to $U(2)$ group. The method clarifies the singular behavior of topologically non-trivial gauge transformations in non-commutative gauge theory, which appears when the gauge transformations are viewed from the commutative gauge theory equivalent to the commutative theory.

1 Introduction

Quantum Hall effect (QHE) is the phenomenon which occurs in the 1+2 dimensional electron fluid under a strong magnetic field applied from outside. It is interesting to note that as is pointed out by Susskind, QHE can be equivalently described using a non-commutative Chern-Simons (CS) theory or a Matrix model of CS-type [4].

Therefore, the treatment of QHE becomes closer to that of non-commutative gauge theory and string theory in particle physics. In these equivalent descriptions, Laughlin theory with a filling fraction $\frac{1}{n}$ (n =integer) becomes non-commutative CS theory with a CS factor (a coefficient of CS action) being n .

In the original system of electron fluid, the fluid becomes incompressible due to the Pauli principle, so that the area occupied by electrons is conserved dynamically. The freedom to change the coordinate system of fluid, by preserving the area, is called Area Preserving Diffeomorphism(APD). The symmetry relating to APD is transferred to the non-commutative $U(1)$ gauge transformation in the non-commutative CS theory, while it becomes in the matrix model of CS-type to be the unitary $U(\infty)$ transformation of the matrix valued coordinates describing electrons.

Quasi-particles and quasi-holes appear as the surplus and deficit of area in the fluid system, or the singularity of the APD. Equivalently, they are described by singular gauge transformations in the non-commutative CS theory, since APD and gauge transformation are equivalent symmetries in different descriptions. In this manner, we can understand the importance of studying singular gauge transformations in non-commutative gauge theory.

Recently, in order to study the exciton state having both quasi-electron and quasi-hole, the usual one matrix model is found not useful in order to give the exciton state, but two matrix model is to be introduced [6].

In this two matrix model, exciton solution is obtained and its dispersion relation is estimated. The estimated dispersion relation shows a stable point at which the distance between hole and electron takes a fixed value. This suggests a possible phase transition from the fluid to the Wigner crystal in the QHE system.

In these treatments the matrix describing particle or hole is infinite dimensional, but Polychronakos has proposed another model of QHE using finite matrix [5].

As was stated above, we have to introduce two matrix model in some case,

or equivalently $U(1)$ gauge field should be duplicated there. In this respect it is interesting to study $U(1) \times U(1)$ non-commutative CS theory, or more generally non-commutative $U(2)$ CS theory. Singular gauge transformations in $U(2)$ model may be related to the solitonic states such as quasi-particle, quasi-hole and exciton.

The purpose of this paper is to generalize the method developed by Polychronakos in studying singular gauge transformation in non-commutative gauge theory. His method uses the Seiberg-Witten map [1]. This mapping relates gauge fields and gauge transformations having different non-commutative parameters θ . In QHE the non-commutative parameter θ is inversely proportional to the filling fraction ν as follows:

$$\theta = \frac{1}{2\pi\rho} = \frac{1}{eB\nu}, \quad (1)$$

where ρ , e , and B are, respectively, the density, the charge of electron, and the magnetic field applied from outside. Therefore, if the magnetic field is kept constant, Seiberg-Witten map may describe the change of quasi-particle, quasi-hole, and exciton states under the change of the filling fraction ν . Therefore, the method may be useful to study the transition of states in Quantum Hall effect occurring when the filling fraction is changed.

We first review the work by Polychronakos on singular gauge transformation in non-commutative CS theory with $U(1)$ gauge group. Next, we generalize it to the same theory with $U(2)$ gauge group. Discussions are prepared finally.

2 Operator formalism of Seiberg-Witten map and singular gauge transformation

First, we give a brief explanation on Seiberg-Witten (SW) map and Seiberg-Witten (SW) equation [1]. The SW map is the expression of gauge field \hat{A} and gauge transformation parameter $\hat{\lambda}$ in a non-commutative gauge theory in terms of those, A and λ , in a commutative gauge theory, namely,

$$\hat{A} = \hat{A}(A), \quad (2)$$

$$\hat{\lambda} = \hat{\lambda}(\lambda). \quad (3)$$

Then, the following consistency condition is naturally imposed:

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}(A) = \hat{A}(A + \delta_{\lambda}A), \quad (4)$$

where $\hat{\delta}_\lambda$ and δ_λ are gauge transformations of the non-commutative theory and the commutative theory, respectively. From this condition, SW map is determined.

The operator formalism of SW map is given originally by Kraus and Shigemori [2]. Polychronakos modified it so that the SW map may preserve the hermiticity [3], which we will review in the following.

We consider D dimensional space, in which $2n$ dimensions are non-commutative coordinates, satisfying

$$[x^\alpha, x^\beta] = i\theta^{\alpha\beta}, \quad (5)$$

while the remaining $D - 2n$ dimensions are commutative coordinates. The middle Greece indices, $(\mu, \nu, \dots = 1, \dots, D)$ denote all dimensions, early Greece indices, $(\alpha, \beta, \dots = 1, \dots, 2n)$ denote non-commutative dimensions, and $(i, j, \dots = 2n + 1, \dots, D)$ denote commutative dimensions. Then, consider $U(N)$ gauge theory, where the gauge field A_μ is $N \times N$ hermitian matrices.

We will start to explain the operator formalism of SW equation. If we obtain SW maps using Eq.(4) for two different theories with two different non-commutative parameters differing infinitesimally by $\delta\theta^{\alpha\beta}$, then the non-commutative gauge fields $A(\theta + \delta\theta)$ and $A(\theta)$ in two different theories are related, giving

$$\delta A_\mu = A_\mu(\theta + \delta\theta) - A_\mu(\theta) = \frac{1}{4}\delta\theta^{\alpha\beta}\{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\}, \quad (6)$$

where the products of fields are understood to be the usual star products, and the field strength is given as usual,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \quad (7)$$

From (6), we can derive the changes of $F_{\mu\nu}$ and gauge transformation parameter λ similarly as in A_μ . These equations giving variation of fields and parameters under the infinitesimal change of θ is called SW equations.

In the following, we always work with the covariant derivatives, $D_\mu = i\partial_\mu + A_\mu$. The derivative in the commutative direction is as usual, but that in the non-commutative direction should be treated carefully.

First, we rewrite the simple derivative in the non-commutative direction as

$$i\partial_\alpha = \omega_{\alpha\beta}x^\beta, \quad (8)$$

This can be understood easily, since

$$[i\partial_\alpha, x^\beta] = i\delta_\alpha^\beta, \quad (9)$$

holds if $\omega_{\alpha\beta}$ is the inverse 2-form of $\omega_{\alpha\beta}$, namely,

$$\omega_{\alpha\beta}\theta^{\beta\gamma} = \delta_{\alpha}^{\gamma}. \quad (10)$$

Important point here is that the simple derivative, $i\partial_{\alpha}$, is not a usual commutative operator but is the non-commutative operator.

Hence, the covariant derivative in the non-commutative direction reads

$$D_{\alpha} = i\partial_{\alpha} + A_{\alpha} = \omega_{\alpha\beta}x^{\beta} + A_{\alpha} \quad , \quad (11)$$

and the field strength satisfies the following equation being different from the usual one,

$$iF_{\alpha\beta} = [D_{\alpha}, D_{\beta}] + i\omega_{\alpha\beta} \quad (12)$$

Now, SW equation for the covariant derivative needs

$$\delta D_{\mu} = -\omega_{\mu\nu}\delta\theta^{\nu\rho}D_{\rho} + \frac{i}{4}\delta\theta^{\alpha\beta}\{D_{\alpha}, [D_{\beta}, D_{\mu}]\} + i[\delta G, D_{\mu}], \quad (13)$$

where δG is an non-covariant object, given by

$$\delta G(D) = \frac{1}{4}\delta\theta^{\alpha\beta}\{i\partial_{\alpha}, D_{\beta}\}. \quad (14)$$

The last term in (13) with $\delta G(D)$ can be regarded as an infinitesimal gauge transformation. As is easily understood, the difference of two covariant objects (covariant derivatives) defined in two different theories are not gauge invariant in both theories, so that the difference may be generated by a gauge transformation, $\delta G(D)$, of the covariant object (covariant derivative) in one of two theories.

Next, we study singular gauge transformation and its topology. Gauge transformation of the covariant derivative by a unitary transformation $U(\theta)$ is as follows:

$$D_{\alpha}(\theta) = U(\theta)^{-1}D_{\alpha}^{(0)}(\theta)U(\theta), \quad (15)$$

where we write θ explicitly, in order to specify the theory. The variation of this equation for an infinitesimal change of θ is

$$\delta D_{\mu}(\theta) = U(\theta)^{-1}(\delta D_{\mu}^{(0)}(\theta))U(\theta) + (\delta U(\theta)^{-1})D_{\mu}^{(0)}(\theta)U(\theta) + U(\theta)^{-1}D_{\mu}^{(0)}(\theta)(\delta U(\theta)) \quad (16)$$

Applying the SW equation for covariant derivatives in (16), we obtain the following result:

$$iU(\theta)^{-1}\delta U(\theta) = \delta G\left(U(\theta)^{-1}D(\theta)U(\theta)\right) - U(\theta)^{-1}\delta G(U(\theta))U(\theta). \quad (17)$$

If we start from the vacuum having $A^{(0)} = 0$, we have

$$D_\mu^{(0)}(\theta) = i\partial_\mu, \quad (18)$$

$$D_\mu(\theta) = U(\theta)^{-1}i\partial_\mu U(\theta) = A_\mu(\theta)', \quad (19)$$

so that the covariant derivative $D_\mu(\theta)$ is identical to a solitonic gauge field obtained from the vacuum by a singular gauge transformation $U(\theta)$ in non-commutative gauge theory.

Here, we review an example of the singular gauge transformation given by Polychronakos in 1+2 dimensional $U(1)$ non-commutative gauge theory in which we have a commutative time, t , and non-commutative space coordinates, x^1 and x^2 . To represent the non-commutativity of the space it is useful to introduce the annihilation and creation operators, a and a^\dagger , by

$$a = \frac{x^1 + ix^2}{\sqrt{2\theta}}, \quad [a, a^\dagger] = 1, \quad (20)$$

and prepare a vacuum $|0\rangle$ satisfying $a|0\rangle = 0$ and the Fock states $|n\rangle (n = 1, 2, \dots)$. Then the covariant coordinates X^1 and X^2 can be defined by

$$Z = \frac{X^1 + iX^2}{\sqrt{2\theta}} = U^{-1}aU, \quad (21)$$

which can be understood from Eqs. (18) and (19), since the covariant coordinates introduced here are the conjugate variables of the covariant derivatives.

Rewriting (14) in terms of covariant coordinates, we obtain

$$i\delta G = \frac{i}{4}\delta\omega_{\alpha\beta}\{x^\alpha, X^\beta\} = \frac{\delta\theta}{4\theta}(\{a^\dagger, Z\} - \{a, Z^\dagger\}). \quad (22)$$

However, there is a freedom to modify the gauge transformation δG within the admissible gauge transformation [5], so we can use instead of δG a simpler gauge transformation $\delta G'$ given by

$$i\delta G' = \frac{\delta\theta}{2\theta}(a^\dagger Z - Z^\dagger a). \quad (23)$$

If we assume that the unitary transformation $U(\theta)$ takes the following form,

$$U(t; \theta) = \sum_{n=0}^{\infty} e^{i\phi_n(t; \theta)} |n\rangle \langle n|, \quad (24)$$

then we have the following differential equation from (17):

$$\delta\phi_n = \frac{\delta\theta}{\theta} n \sin(\phi_n - \phi_{n-1}) \quad . \quad (25)$$

We can solve this equation, starting from a non-commutative theory with non-commutative parameter $\theta = \theta_0$, under the following boundary conditions

$$\phi_0(t = +\infty; \theta_0) - \phi_0(t = -\infty; \theta_0) = 2\pi, \quad (26)$$

$$\phi_n(t; \theta_0) = 0 \quad \text{for } (n = 1, 2, \dots). \quad (27)$$

The dependence of ϕ on n gives the dependence of ϕ on the spacial radius $r = \sqrt{(x^1)^2 + (x^2)^2}$, since we have approximately $r \approx \sqrt{2n\theta}$. Eq.(26) means that a topological excitation is created around $r = 0$.

The solution of the above differential equation shows that the gauge transformation viewed from the commutative theory with $\theta = 0$ gives a singular behavior when ϕ approaches the value π at the origin $r = 0$ [3]. Namely, the value of ϕ at $r = 0$ increases in time and attains π , but then suddenly jumps to $-\pi$ and increases again in time. There appears a kink at $r = 0$ as a function of t , and spacial profile near the kink is a pulse with height π and a pulse with height $-\pi$ before and after the time when ϕ crosses the kink position. This singular behavior is a characteristic of the solitonic solution in non-commutative gauge theory. Figure, (1), (2), (3), and (4) show the spacial profiles of the phase ϕ of singular gauge transformation at time t_1, t_2, t_3 , and t_4 ($t_1 < t_2 < t_3 < t_4$), respectively.

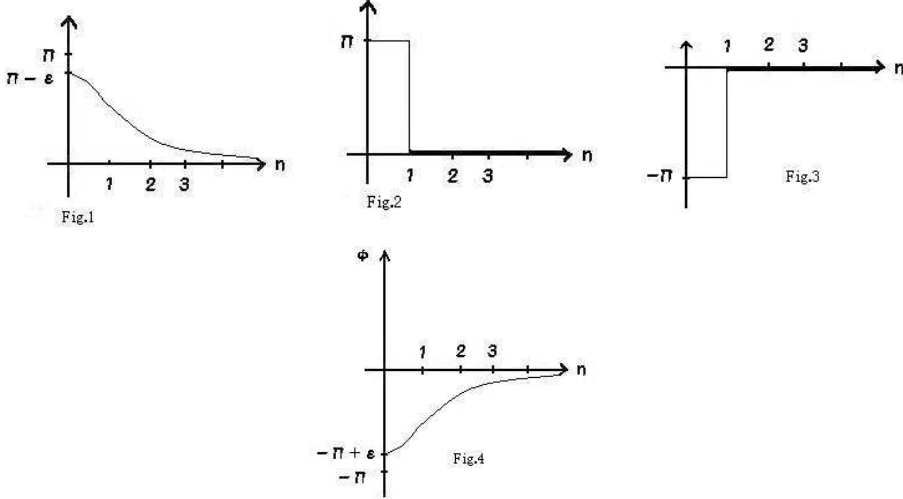


Fig. 1: Figs. (1), (2), (3), and (4) depict the spacial profile (n dependence) of the phase ϕ at $t = t_1, t_2, t_3$, and t_4 ($t_1 < t_2 < t_3 < t_4$), respectively.

3 Generalizaion to U(2) gauge group

In this section, we will generalize the prescription of studying singular gauge transformation to the non-commutative gauge theory with $U(2)$ group. The gauge tranformation should satisfy the SW equation (17) under the cahnge of the non-commutative parameter θ ,

$$iU^{-1}\delta U = \delta G'(U^{-1}DU) - U^{-1}\delta G'(D)U,$$

where we have adopted the simpler choice of $\delta G'$ in (23). Then, we obtain

$$\begin{aligned} iU^{-1}\delta U &= \frac{\delta\theta}{2i\theta}(a^\dagger Z - Z^\dagger a) \\ &= \frac{\delta\theta}{2i\theta}(a^\dagger U^{-1}aU - U^{-1}a^\dagger Ua), \end{aligned} \quad (28)$$

where we have used $\delta G'(D) = 0$.

Now, the SW equation is reduced to

$$\delta U_n^\dagger U_n = \frac{\delta\theta}{2\theta}n(U_{n-1}^\dagger U_n - U_n^\dagger U_{n-1}). \quad (29)$$

The $U(2)$ gauge transformation is given by

$$U(\theta) = \sum_{n=0}^{\infty} U_n |n\rangle \langle n|, \quad (30)$$

$$U_n(\theta) = \sum_{n=0}^{\infty} e^{i\phi_n} \begin{pmatrix} \alpha_n & \beta_n \\ -\beta_n^* & \alpha_n^* \end{pmatrix}, \quad (31)$$

where $|\alpha_n|^2 + |\beta_n|^2 = 1$, and therefore $\delta\alpha_n\alpha_n^* + \alpha_n\delta\alpha_n^* + \delta\beta_n\beta_n^* + \beta_n\delta\beta_n^* = 0$ should hold for the $U(2)$ transformation.

Then, the SW equations read

$$\delta\phi_n = -\frac{\delta\theta}{2\theta}n \sin(\phi_n - \phi_{n-1}) (\alpha_n^* \alpha_{n-1} + \beta_n \beta_{n-1}^* + (h.c.)), \quad (32)$$

$$\delta\alpha_n = -\frac{\delta\theta}{2\theta}n 2 \cos(\phi_n - \phi_{n-1})(\alpha_n - \alpha_{n-1}), \quad (33)$$

$$\delta\beta_n = -\frac{\delta\theta}{2\theta}n 2 \cos(\phi_n - \phi_{n-1})(\beta_n - \beta_{n-1}). \quad (34)$$

If we express α_n and β_n in terms of three phases, ϑ_n , ψ_n , and χ_n , the $U(2)$ property is manifestly guranteed. That is, we use the following parametrization:

$$\alpha_n \equiv e^{i\psi_n} \sin \vartheta_n, \quad (35)$$

$$\beta_n \equiv e^{i\chi_n} \cos \vartheta_n. \quad (36)$$

Now, we have the SW equations as follows:

$$\begin{aligned}\delta\phi_n &= -\frac{\delta\theta}{\theta}n\sin(\phi_n - \phi_{n-1}) \\ &\times (\cos\vartheta_n \cos\vartheta_{n-1} \cos(\chi_n - \chi_{n-1}) + \sin\vartheta_n \sin\vartheta_{n-1} \cos(\psi_n - \psi_{n-1})),\end{aligned}\quad (37)$$

$$\begin{aligned}\delta\vartheta_n &= -\frac{\delta\theta}{\theta}n\cos(\phi_n - \phi_{n-1}) \\ &\times (\sin\vartheta_n \cos\vartheta_{n-1} \cos(\chi_n - \chi_{n-1}) - \cos\vartheta_n \sin\vartheta_{n-1} \cos(\psi_n - \psi_{n-1})),\end{aligned}\quad (38)$$

$$\delta\psi_n = -\frac{\delta\theta}{\theta}n\cos(\phi_n - \phi_{n-1})\left(\frac{\sin\vartheta_{n-1}}{\sin\vartheta_n}\right)\sin(\psi_n - \psi_{n-1}),\quad (39)$$

$$\delta\chi_n = -\frac{\delta\theta}{\theta}n\cos(\phi_n - \phi_{n-1})\left(\frac{\cos\vartheta_{n-1}}{\cos\vartheta_n}\right)\sin(\chi_n - \chi_{n-1}).\quad (40)$$

In general, $\phi_0, \vartheta_0, \psi_0$, and χ_0 for $n = 0$ do not change during the change of θ moving from the original θ_0 to zero. Therefore these phases keep the profiles in the original theory. The phases for $n > 0$ may change, when θ is reduced towards zero, namely $\frac{d\theta}{\theta} < 0$. In the region well apart from the location of pulses, we can consider that the difference of phases at n and $n - 1$ are not large, so that we can set the following approximation:

$$\begin{cases} \phi_n \simeq \phi_{n-1} \\ \vartheta_n \simeq \vartheta_{n-1} \\ \psi_n \simeq \psi_{n-1} \\ \chi_n \simeq \chi_{n-1} \end{cases}\quad (41)$$

Therefore in this region, we have approximately

$$\delta\phi_n \simeq -\frac{\delta\theta}{\theta}n\sin(\phi_n - \phi_{n-1}),\quad (42)$$

$$\delta\vartheta_n \simeq -\frac{\delta\theta}{\theta}n\sin(\vartheta_n - \vartheta_{n-1}),\quad (43)$$

$$\delta\psi_n \simeq -\frac{\delta\theta}{\theta}n\sin(\psi_n - \psi_{n-1}),\quad (44)$$

$$\delta\chi_n \simeq -\frac{\delta\theta}{\theta}n\sin(\chi_n - \chi_{n-1}).\quad (45)$$

These four phases satisfy approximately the same SW equations in general. Hence we can follow the discussion which is given originally by Polychronakos and is reviewed in the last section. If the phases approach π from below, the difference of phases between n and $n - 1$ increase when θ decreases, while the phases exceed π , the difference of the phases decrease (because of the sign change of the sine function). However, all the phases should be zero asymptotically for $n \rightarrow \infty$, or $r \rightarrow \infty$. Therefore, the region of phases exceeding π should be understood as

$-\pi$ by periodicity, so that even if the difference of phases between n and $n - 1$ decrease, the phases can be asymptotically zero for $n \rightarrow \infty$. Roughly speaking, the asymptotic behavior of phases for $n \rightarrow \infty$ can be expressed in terms of the differential equations:

$$\left(\frac{\partial}{\partial \ln \theta} \pm \frac{\partial}{\partial \ln n} \right) (\phi, \vartheta, \psi, \xi) = 0, \quad (46)$$

where \pm sign is the remnant of the original sign of the sine function. The $+$ sign describes the region of phase difference from 0 to π , and the $-$ sign describes the region of phase difference from $-\pi$ to 0. From this differential equation, all the phases are asymptotically a function of the ratio $\frac{n}{\theta}$ if the phase difference is from 0 to π , while they are function of $n\theta$ if the phase difference is from $-\pi$ to 0. This gives the quantitative behavior of the change of phase when θ decreases.

In the $U(2)$ case, we have four phases, one of which ϕ is the $U(1)$ phase, but the remaining three are $SU(2)$ phases. The same discussion can be applied for three $SU(2)$ phases as in the $U(1)$ phase. Therefore if we denote these phases generally Φ . Then, if Φ is topologically non-trivial,

$$\Phi_0(t = +\infty; \theta_0) - \Phi_0(t = -\infty; \theta_0) = 2\pi, \quad (47)$$

$$\Phi_n(t; \theta_0) = 0 \quad (n = 1, 2, \dots), \quad (48)$$

we have the same singular behavior, having kink and pulse for the $SU(2)$ phases. Here one commutative dimension t and two non-commutative dimensions x^1 and x^2 are treated separately, so that only $U(1)$ -like singular behavior is obtained. Relevant topology here is $\pi_1(U(2))$. To obtain singular behavior specific to $U(2)$, being relevant to the topology of $\pi_2(U(2)/Z_2)$, we have to consider a more general dependence of the phases on t, x^1 and x^2 which is not discussed in this paper.

4 Conclusion

We have studied singular gauge transformations in the non-commutative $U(2)$ gauge theories, by generalizing the method proposed by Polychronakos [3] to study the singular gauge transformation in non-commutative $U(1)$ gauge theories. The space-time dimension of the theory is three in which time coordinate is commutative, but two spacial coordinates are non-commutative with each other. A typical example is the 1+2 dimensional non-commutative Chern-Simons gauge theory applicable to Quantum Hall effects.

The method uses the operator formalism of Seiberg-Witten map [1] which connects different theories with different non-commutative parameters θ . Using this map, the gauge transformation in the non-commutative theory can be viewed from the corresponding commutative theory equivalent to the non commutative theory. In $U(2)$ gauge transformation we have four phases. SW-equations for these four phases are derived explicitly. From these equations, we obtain a similar singular behavior for these phases as is depicted in Figs. (1), (2), (3), and (4).

In order to study the exciton in Quantum Hall effects(QHE), it may be necessary to introduce two matrix model. One matrix model of Chern-Simons(CS) type and the non-commutative Chern-Simons(CS) gauge theory with $U(1)$ group describe equivalently QHE, so that the study of singular gauge transformation in non-commutative $U(2)$ CS gauge theory may be useful for the study of QHE using two matrix model.

It is also interesting to connect the change of the non-commutative parameter θ by Seiberg-Witten map to the change of filling fraction ν , and to study the phase structure of QHE in terms of the filling fraction ν .

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References

- [1] Nathan Seiberg and Edward Witten, "String Theory and Noncommutative Geometry", JHEP 9909, 032 (1999), hep-th/9908142
- [2] P.Kraus and M.Shigemori,"Non-Commutative Instantons and the Seiberg-Witten Map", JHEP 0206, 034 (2002), hep-th/0110035
- [3] Alexios P.Polychronakos, "Seiberg-Witten map and topology", hep-th/0206013
- [4] Leonard.Susskind, "The Quantum Hall Fluid and Non-Commutative Chern Simons Theory", hep-th/0101029
- [5] Alexios P. Polychronakos, "Quantum Hall states as matrix Chern-Simons theory", JHEP 0104, 011 (2001), hep-th/0103013

- [6] Yuko Kobashi, Bhabani Prasad Mandal, Akio Sugamoto, "Exciton in Matrix Formulation of Quantum Hall Effect", Nucl. Phys.B 679 [FS] (2004) 405-426, hep-th/0202050.